J. Fluid Mech. (1973), *vol.* 58, *part* 4, *pp.* 763–775
Printed in Great Britain

Perturbation methods in boundary-layer theory

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(Received 21 August 1972)

To solve a mathematical problem involving a small parameter, it is customary to expand the solution in powers of that parameter. In singular cases the resultant linearized problem may be insoluble, and in some such cases it is appropriate to expand the solution in powers of the *square-root* of the small parameter. These cases are associated with bifurcation of the solution. The method is illustrated by applying it to the Falkner–Skan equation and to a problem in hydrodynamic instability. In particular, Hartree's conjecture, that near separation the skin friction varies like the square-root of the appropriate parameter of the Falkner–Skan equation, is substantiated.

1. Introduction

The literature of boundary-layer theory recounts many occasions where problems are linearized as part of some perturbation procedure. Here we shall treat a special case of such a perturbation procedure in which a naïve expansion of the solution in powers of the natural small parameter is invalid, although linearization of the boundary-layer problem about a particular solution is desirable and the correct expansion of the solution is in powers of the *square-root* of the small parameter. The ideas stem from the rigorous background of the Liapounov–Schmidt theory of bifurcation and its developments such as Leray–Schauder theory (cf. Stakgold 1971). The method is useful in applied mathematics at large, and is worthy of more use. Some applications to fluid mechanics are suggested by examples in the following sections. It seems easiest to explain the basic idea by considering a simple classic example, that of the Falkner–Skan equation.

2. Perturbation of solutions of the Falkner–Skan equation

The Falkner–Skan equation

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{1}$$

and its boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1 \tag{2}$$

are well known (cf. Rosenhead 1963, chap. 5) to have solutions $f(\eta)$ for $0 \leqslant \eta < \infty$, $\beta_* \leqslant \beta \leqslant 2$. Here it should be understood that the condition at infinity means that $1 - f'(\eta)$ becomes exponentially small as $\eta \to \infty$, and that β_* is the value of β

for which $f''(0) = 0$, viz. $\beta_* = -0{\cdot}19883774$. It is known that there is one solution satisfying $0 < f'(\eta) < 1$ and $f''(\eta) > 0$ for $0 < \eta < \infty$ for each value of β in the range $\beta_* \leqslant \beta \leqslant 2$, but that there is a second solution satisfying $f'(\eta) < 1$ for $0 < \eta < \infty$ and $f'(\eta) < 0$ for fairly small η when β lies in the range $\beta_* < \beta < 0$. The latter solution represents a boundary layer with a region of reversed flow near the wall $\eta = 0$.

Following Rubbert & Landahl (1967), we start by supposing that we know $f(\eta) = f_0(\eta)$ for $\beta = \beta_0$, where β_0 is any given number such that $\beta_* \leqslant \beta_0 \leqslant 2$, and seek $f(\eta)$ for neighbouring values of β. Here $f_0(\eta)$ may be either of the two solutions, with or without reversed flow. Then it is natural to try to expand

$$f(\eta) = f_0(\eta) + (\beta-\beta_0) f_1(\eta) + (\beta-\beta_0)^2 f_2(\eta) + \ldots \tag{3}$$

for small values of $\beta-\beta_0$. On equating coefficients of powers of the small parameter $\beta-\beta_0$ in the Falkner–Skan problem this formally gives us a succession of linear inhomogeneous problems to solve:

$$\left.\begin{aligned} Lf_1 &\equiv f_1''' + f_0 f_1'' - 2\beta_0 f_0' f_1' + f_0'' f_1 = -(1-f_0'^2),\\ Bf_1 &\equiv \begin{pmatrix} f_1(0) \\ f_1'(0) \\ f_1'(\infty) \end{pmatrix} = 0; \end{aligned}\right\} \tag{P_1}$$

$$Lf_2 = \beta_0 f_1'^2 - f_1 f_1'' + 2 f_0' f_1', \quad Bf_2 = 0; \tag{P_2}$$

etc. Thus we suppose that the solution $f_0(\eta)$ of the nonlinear problem (1) and (2) with $\beta = \beta_0$ is known completely, and seek to solve the linear problems (P_1), (P_2), etc.

Well-known properties of ordinary differential equations (cf. Friedman 1956, chap. 3) imply the Fredholm alternatives for problems (P_n) $(n = 1, 2, \ldots)$ that either

(i) the homogeneous problem H, namely

$$Lh = 0, \quad Bh = 0, \tag{H}$$

has no non-trivial solution and (P_n) has a unique solution; or

(ii) (H) has a non-trivial solution h, the adjoint homogeneous problem $(H^\dagger)$ has a non-trivial solution $h^\dagger$, and either (*a*) (P_n) has an infinity of solutions, if its right-hand side is orthogonal to $h^\dagger$, or (*b*) (P_n) has no solution, if its right-hand side is not orthogonal to $h^\dagger$.

Chen & Libby (1968) have shown that the problem (H) has no non-trivial solution h when $\beta_* < \beta_0 \leqslant 2$. Therefore, under alternative (i), it is possible to solve problem P_1 by the method of variation of parameters. This holds in principle, though in practice we do not know three independent solutions of the equation $Lg = 0$ in terms of elementary functions, and so have to resort to numerical methods to find f_1. In the same way we can solve (P_2), (P_3), etc. and proceed with a formal solution to as high a degree of approximation as we wish.

A special case arises when $\beta_0 = \beta_*$. Then it is well known that a solution to (H) is simply

$$h = f_0'(\eta), \tag{4}$$

because $Lf_0' = 0$ (for all β_0 incidentally) and $Bf_0' = 0$ because $f_0''(0) = 0$ when $\beta_0 = \beta_*$. Therefore a non-trivial solution h of the homogeneous problem exists in this special case, and alternative (ii) applies.

It is now necessary to digress by solving the adjoint problem,

$$L^\dagger h^\dagger = 0, \quad B^\dagger h^\dagger = 0. \tag{$H^\dagger$}$$

Here we must find $L^\dagger$ and $B^\dagger$ explicitly so that the adjoint relation

$$\int_0^\infty g L^\dagger g^\dagger \, d\eta = \int_0^\infty g^\dagger \, Lg \, d\eta \tag{5}$$

holds for all well-behaved functions g and $g^\dagger$ satisfying the boundary conditions $Bg = 0$ and $B^\dagger g^\dagger = 0$. Now integration by parts gives

$$\int_0^\infty g^\dagger Lg \, d\eta = \int_0^\infty g L^\dagger g^\dagger \, d\eta + [g^\dagger g'' - (g^{\dagger\prime} - f_0 g^\dagger) g' + (g^{\dagger\prime\prime} - f_0 g^{\dagger\prime} - (1+2\beta_*) f_0' g^\dagger) g]_0^\infty, \tag{6}$$

where the differential operator $L^\dagger$ is defined so that

$$L^\dagger g^\dagger \equiv -g^{\dagger\prime\prime\prime} + f_0 g^{\dagger\prime\prime} + 2(1+\beta_*)(f_0' g^\dagger)'. \tag{7}$$

It follows that we must define the operator $B^\dagger$ so that

$$B^\dagger g^\dagger \equiv \begin{pmatrix} g^\dagger(0) \\ [(g^{\dagger\prime} - f_0 g^\dagger)\,\mathrm{A}]_{\eta=\infty} \\ [g^{\dagger\prime\prime} - f_0 g^{\dagger\prime} - (1+2\beta_*) f_0' g^\dagger]_{\eta=\infty} \end{pmatrix}. \tag{8}$$

Here A denotes any exponentially decreasing function of η, the second boundary condition meaning that $g^{\dagger\prime} - f_0 g^\dagger$ does *not* increase exponentially at infinity. The solution $h = f_0'$ of $Lh = 0$ gives an integration factor of $L^\dagger h^\dagger$; in fact

$$f_0' L^\dagger h^\dagger = d[-f_0' h^{\dagger\prime\prime} + (f_0 f_0' + f_0'') h^{\dagger\prime} + \{\beta_* + (1+\beta_*) f_0'^2\} h^\dagger]/d\eta. \tag{9}$$

Therefore

$$-f_0' h^{\dagger\prime\prime} + (f_0 f_0' + f_0'') h^{\dagger\prime} + \{\beta_* + (1+\beta_*) f_0'^2\} h^\dagger = 0, \tag{10}$$

on use of the conditions $B^\dagger h^\dagger = 0$ at infinity and of the properties that $f_0'(\infty) = 1$ and $f_0''(\infty) = 0$. We have integrated system ($H^\dagger$) numerically to find $h^\dagger$ when $\beta_0 = \beta_*$. With the normalization that $h^{\dagger\prime}(0) = 1$, we find that $h^\dagger(\eta) > 0$ for $0 < \eta < \infty$.

After this digression on the adjoint problem, we return to problem (P_1), when $\beta_0 = \beta_*$. Equations (P_1), (5) and ($H^\dagger$) give

$$0 = -\int_0^\infty f_1 L^\dagger h^\dagger \, d\eta = -\int_0^\infty h^\dagger L f_1 \, d\eta,$$

$$= \int_0^\infty h^\dagger (1 - f_0'^2) \, d\eta \equiv I_1, \text{ say}, \tag{11}$$

on the assumption that f_1 exists. But we already know that $h^\dagger(\eta) > 0$ and $0 < f_0'(\eta) < 1$ for $0 < \eta < \infty$. Therefore I_1 cannot vanish, and f_1 does not exist when $\beta_0 = \beta_*$.

The whole perturbation expansion (3) is seen to break down in this special case $\beta_0 = \beta_*$ for which non-trivial solutions h and $h^\dagger$ exist. The cure for the trouble is simply to expand $f(\eta)$ in the form

$$f(\eta) = f_0(\eta) + (\beta-\beta_*)^{\frac{1}{2}} f_{\frac{1}{2}}(\eta) + (\beta-\beta_*) f_1(\eta) + \ldots \tag{12}$$

and equate coefficients of powers of $(\beta-\beta_*)^{\frac{1}{2}}$ in the Falkner–Skan problem (1) and (2). This time we find that

$$Lf_{\frac{1}{2}} = 0, \quad Bf_{\frac{1}{2}} = 0; \tag{$Q_{\frac{1}{2}}$}$$

$$Lf_1 = \beta_* f_{\frac{1}{2}}'^2 - f_{\frac{1}{2}} f_{\frac{1}{2}}'' - (1-f_0'^2), \quad Bf_1 = 0; \tag{Q_1}$$

etc. The solution of problem $(Q_{\frac{1}{2}})$ is simply

$$f_{\frac{1}{2}} = af_0', \tag{13}$$

where a is an arbitrary constant, because $(Q_{\frac{1}{2}})$ is the same problem as (H). We shall determine the constant a by requiring that the problem (Q_1) be soluble. Now (Q_1) gives

$$Lf_1 = a^2(\beta_* f_0''^2 - f_0' f_0''') - (1-f_0'^2). \tag{14}$$

Therefore the adjoint relation for (Q_1) becomes

$$0 = \int_0^\infty f_1 L^\dagger h^\dagger \, d\eta = \int_0^\infty h^\dagger L f_1 \, d\eta = a^2 I_2 - I_1,$$

where

$$I_2 \equiv \int_0^\infty h^\dagger(\beta_* f_0''^2 - f_0' f_0''')\, d\eta$$

$$= \int_0^\infty h^\dagger \{\beta_* f_0''^2 + f_0 f_0' f_0'' + \beta_*(1-f_0'^2) f_0'\}\, d\eta, \tag{15}$$

and I_1 is given in (11). Therefore

$$a = \pm (I_1/I_2)^{\frac{1}{2}}, \tag{16}$$

provided that $I_2 \neq 0$. (If I_2 were zero, we would next try to expand the solution in powers of $(\beta-\beta_*)^{\frac{1}{3}}$.) With this proviso, we have a formal solution f_1 from which we may proceed to find f_2 etc. by similar methods. This gives two solutions,

$$f(\eta) = f_0(\eta) \pm \{(\beta-\beta_*) I_1/I_2\}^{\frac{1}{2}} f_0'(\eta) + O(\beta-\beta_*), \tag{17}$$

as $\beta \to \beta_* + 0$.

This analytic result gives the qualitative behaviour of the graph of $f''(0)$ against β for values of β near β_* as found by numerical work of Hartree (cf. Rosenhead 1963, chap. 5). In particular, it gives *two* branches of the graph near $\beta = \beta_*$, reversed flow corresponding to points on the lower branch with the negative square-root. (If I_1/I_2 happened to be negative, then a would be pure imaginary and it would be more appropriate to write $f = f_0 \pm \{-(\beta_*-\beta) I_1/I_2\}^{\frac{1}{2}} f_0'$ for $0 < \beta_* - \beta \ll 1$.) Our analytic results seem to be new, though they only substantiate and go a little beyond some conjectures Hartree made on the basis of his numerical work.

We computed $h^\dagger$ with the normalization $h^\dagger(0) = 1$ and then integrated to find $I_1 = 1{\cdot}7936$ and $I_2 = 0{\cdot}099419$. This gives

$$I_1/I_2 = 18{\cdot}04. \tag{18}$$

(We used the 'shooting' method to find $h^\dagger$. No difficulty was encountered in the numerical work. In particular, although $h^\dagger \sim \text{constant} \times \eta^{-(1+2\beta_*)}$ as $\eta \to \infty$, no long range of integration was necessary because (*a*) the integrands of I_1 and I_2 decay exponentially, and (*b*) $h^{\dagger}$ was insensitive to precise imposition of the asymptotic boundary condition.) We verified the results (17) and (18) by direct numerical integration of the two solutions of the Falkner–Skan problem (1) and (2) for $\beta = -0{\cdot}19880$ and $-0{\cdot}19882$. A parabolic fit of the graph of $f''(0)$ against β through the point $f''(0) = 0$, $\beta = \beta_*$ and the two points corresponding to the values of $f''(0)$ at each neighbouring value of β agrees with our results (17) and (18).

3. General application of method

Looking back over the above example, we see that expansion in powers of $\beta-\beta_0$ is valid in general, but not in particular when $\beta_0 = \beta_*$. In other words, the parametric derivative $\partial f(\eta,\beta)/\partial\beta$ of the solution of the Falkner–Skan problem exists, except when $\beta = \beta_*$. Recognition of this trouble and knowledge of its cure supplements Rubbert & Landahl's (1967) discussion of the wide use of parametric differentiation of nonlinear solutions in fluid mechanics. In our example we have only used ideas that are common to applied mathematicians, but they deserve more use, as the examples of the following sections show. These examples involve ordinary differential equations of low order, though the basic idea may be used for ordinary differential equations of high order, for partial differential equations, for difference equations etc. We shall emphasize applications to boundary-layer theory, because it seems there are many new and useful ones.

Chen & Libby (1968) considered the variation downstream of a steady boundary layer which was slightly different from the Falkner–Skan solution at a given station. They were led to inhomogeneous linear problems of the form of (P_n), which they solved in general, but not when $\beta = \beta_*$. They also discovered the interesting result that the solutions corresponding to reversed flow are unstable but that other solutions are stable, i.e. in steady flow in the boundary layer with flow reversed a small irregularity grows downstream but in a unidirectional boundary layer a small irregularity decays. Thus $\beta = \beta_*$ is a margin of stability between the stable 'upper branch' marked by unidirectional flow and the upper sign of a in (16), and the unstable 'lower branch' marked by reversed flow and the lower sign of a. One may conjecture that the bifurcation of the solution of the Falkner–Skan equation at $\beta = \beta_*$ is related to the margin of stability found by Chen & Libby (1968).

Kelly (1962) considered the development in time of small perturbations of the Falkner–Skan solution. This again leads to linearized equations similar to those of § 2. Kelly found stability in this sense for $\beta = 1$ and $\beta = \frac{1}{2}$ but did not consider other values of β.

In finding approximations to the solution of the Navier–Stokes equations higher than that of the Falkner–Skan solution, inhomogeneous linear problems recur (cf. Van Dyke 1964, §§ 7.8, 7.11) and the linear analysis discussed in § 2 is relevant.

It is customary to regard the Falkner–Skan solution as a crude local approximation to that of the boundary layer on a curved body and to allow β to vary to simulate the development of the boundary layer downstream. Thus, if a bluff body is placed in a uniform stream, $\beta = 1$ at the forward stagnation point and β decreases around the body. The boundary layer separates where β decreases to β_*, so that the approach to stagnation is determined by the special rather than the general case of § 2. (However, if separation occurs the reversed flow invalidates the derivation of the Falkner–Skan equation as an asymptotic approximation for the Navier–Stokes equations.) Narasimha & Ojha (1967) and Cooke (1966), in an unpublished report, have given a more refined theory, considering a wall with small curvature. Again, they found inhomogeneous linear problems of the same form as (P_n) and solved them only for $\beta > \beta_*$. The solution for $\beta = \beta_*$ is found in § 5 of this paper.

These problems of perturbations of the Falkner–Skan equation have analogies throughout boundary-layer theory. Bifurcation of dual solutions is known to occur in many problems: boundary-layer flows of compressible fluid (cf. Stewartson 1964, chap. 4); axisymmetric flow between two parallel rotating disks (Mellor, Chapple & Stokes 1968); free-convection flow at a saddle point (in unpublished work following Banks 1972); magnetohydrodynamic boundary layers (Stewartson & Wilson 1964). Bifurcation may occur for free convection between vertical walls (Watson & Poots 1971). We believe that the ideas of linear analysis used in § 2 may be helpful in these other problems, although the details will differ. In particular, the linearized differential system will usually be of order higher than three, and a solution of the homogeneous problem may be obtainable only by numerical analysis. Our finding a third-order equation for (H) and an explicit solution $h = f_0'$ which also satisfied the boundary conditions when $\beta_0 = \beta_*$ simplified, but was not essential to, the analysis of § 2. Most boundary-layer equations do not explicitly depend upon η, the scaled distance from the wall, and so it follows that the derivative of the solution will satisfy the homogeneous linearized equation; however, the Falkner–Skan problem is exceptional in that $h = f_0'$ satisfies the homogeneous boundary conditions as well when $\beta_0 = \beta_*$.

Dual solutions are also known for forced flow at a point of attachment (Schofield & Davey 1967), but bifurcation may or may not be related to the ideas of § 2. We feel that the coincidence of linear marginal stability and bifurcation of non-linear solutions is important, as indeed was shown by Liapounov in his original study of the stability of rotating fluid masses.

4. A three-dimensional boundary layer on a wedge with a weak cross-flow

We shall now examine an example in detail to substantiate the discussion of the last section. Wilkinson (1954) considered a three-dimensional boundary layer near a rigid wall $z = 0$ having an outer flow with velocity $x^m\mathbf{i} + \alpha y\mathbf{j}$, after the manner of Howarth (1951). Wilkinson took $\alpha > 0$ and assumed that the tangential velocity components within the layer could be expanded in the forms

$$\left.\begin{aligned} u &= x^m f_0'(\eta) + \alpha x f_1'(\eta) + O(\alpha^2), \\ v &= \qquad\qquad \alpha y g_1'(\eta) + O(\alpha^2), \end{aligned}\right\} \tag{19}$$

as $\alpha \to 0$ for fixed $\beta = 2m/(m+1)$, where $\eta \equiv [(m+1)/2\nu]^{\frac{1}{2}} x^{\frac{1}{2}(m-1)} z$ is the usual scaled distance from the wall. He then showed that f_0 is the usual solution of the Falkner–Skan problem (1) and (2); that

$$\left.\begin{aligned} g_1''' + f_0 g_1'' &= 0, \\ g_1(0) = g_1'(0) = 0, \quad g_1'(\infty) &= 1; \end{aligned}\right\} \tag{20}$$

and that

$$\left.\begin{aligned} L_1 f_1 &\equiv f_1''' + f_0 f_1'' - 2f_0' f_1' + (3-2\beta) f_0'' f_1 \\ &= -(2-\beta) f_0'' g_1, \\ Bf_1 &= 0. \end{aligned}\right\} \tag{21}$$

Explicit integration of the system (20) gives

$$g_1(\eta) = \eta - \frac{\int_0^{\eta} \int_{\eta_3}^{\infty} \exp\left\{-\int_0^{\eta_2} f_0(\eta_1)\, d\eta_1\right\} d\eta_2\, d\eta_3}{\int_0^{\infty} \exp\left\{-\int_0^{\eta_2} f_0(\eta_1)\, d\eta_1\right\} d\eta_2}.$$

Wilkinson (1954, p. 76) found f_1 numerically and noted that his solution broke down when $\beta = \beta_*$ but did not proceed further with it. In the light of our previous work, we see at once that $L_1 f_0' = 0$ and $Bf_0' = 0$ when $\beta = \beta_*$, so the nature of the trouble and its cure can be readily anticipated.

Since $\alpha > 0$ we expand u and v in the forms

$$\left.\begin{aligned} u &= x^m f_0'(\eta) + \alpha^{\frac{1}{2}} x^{\frac{1}{2}(m+1)} f_{\frac{1}{2}}'(\eta) + \alpha x f_1'(\eta) + O(\alpha^{\frac{3}{2}}), \\ v &= \alpha y g_1'(\eta) + O(\alpha^{\frac{3}{2}}), \end{aligned}\right\} \tag{22}$$

as $\alpha \to 0+$ for $\beta = \beta_*$, and equate coefficients of powers of $\alpha^{\frac{1}{2}}$ in the boundary-layer equations. After a little manipulation we find that

$$\left.\begin{aligned} L_{\frac{1}{2}} f_{\frac{1}{2}} \equiv f_{\frac{1}{2}}''' + f_0 f_{\frac{1}{2}}'' - (1+\beta_*) f_0' f_{\frac{1}{2}}' + (2-\beta_*) f_0'' f_{\frac{1}{2}} &= 0, \\ Bf_{\frac{1}{2}} &= 0; \end{aligned}\right\} \tag{23}$$

and that

$$\left.\begin{aligned} L_1 f_1 &= -(2-\beta_*) f_0'' g_1 + f_{\frac{1}{2}}'^2 - (2-\beta_*) f_{\frac{1}{2}} f_{\frac{1}{2}}'', \\ Bf_1 &= 0. \end{aligned}\right\} \tag{24}$$

By inspection we see that

$$f_{\frac{1}{2}} = b f_0' \tag{25}$$

is the solution of problem (23) for an arbitrary constant b, whence (24) becomes

$$\left.\begin{aligned} L_1 f_1 &= -(2-\beta_*) f_0'' g_1 + b^2\{f_0''^2 + (2-\beta_*) f_0 f_0' f_0'' + \beta_*(2-\beta_*) f_0'(1-f_0'^2)\}, \\ Bf_1 &= 0. \end{aligned}\right\} \tag{26}$$

To find b from the solubility condition of problem (26), we must consider the solution of the adjoint homogeneous problem

$$L_1^\dagger h_1^\dagger = 0, \quad B_1^\dagger h_1^\dagger = 0. \tag{27}$$

Now integration by parts gives

$$\int_0^\infty g^\dagger L_1 g\, d\eta = \int_0^\infty g L_1^\dagger g^\dagger\, d\eta + [g^\dagger g'' - (g^{\dagger\prime} - f_0 g^\dagger) g' + (g^{\dagger\prime\prime} - f_0 g^{\dagger\prime} - 3f_0' g^\dagger) g]_0^\infty, \tag{28}$$

where

$$L_1^\dagger g^\dagger \equiv -g^{\dagger\prime\prime\prime} + (f_0 g^\dagger)'' + 2(f_0' g^\dagger)' + (3-2\beta_*) f_0'' g^\dagger. \tag{29}$$

Therefore
$$B_1^\dagger g^\dagger \equiv \begin{pmatrix} g^\dagger(0) \\ [\mathrm{A}(g^{\dagger\prime} - f_0 g^\dagger)]_{\eta=\infty} \\ [g^{\dagger\prime\prime} - f_0 g^{\dagger\prime} - 3f_0' g^\dagger]_{\eta=\infty} \end{pmatrix}. \tag{30}$$

By the methods of § 2 one may now show that
$$-f_0' h_1^{\dagger\prime\prime} + (f_0'' + f_0 f_0') h_1^{\dagger\prime} + \{\beta_* + (3-\beta_*) f_0'^2\} h_1^\dagger = 0. \tag{31}$$

Therefore we find that
$$0 = \int_0^\infty f_1 L_1^\dagger h_1^\dagger \, d\eta = \int_0^\infty h_1^\dagger L_1 f_1 \, d\eta$$
$$= -(2-\beta_*) J_1 + b^2 J_2, \tag{32}$$

where
$$J_1 \equiv \int_0^\infty f_0'' g_1 h_1^\dagger \, d\eta = 0{\cdot}19012, \tag{33}$$

$$J_2 \equiv \int_0^\infty \{f_0''^2 + (2-\beta_*) f_0 f_0' f_0'' + \beta_*(2-\beta_*) f_0'(1-f_0'^2)\} h_1^\dagger \, d\eta = 0{\cdot}099419. \tag{34}$$

These results give
$$b = \pm\{(2-\beta_*) J_1/J_2\}^{\frac{1}{2}} = \pm 2{\cdot}051. \tag{35}$$

Note that (22), (25) and (35) give *two* solutions for $\beta = \beta_*$. For $\alpha x^{(1-m)} \ll 1$, equation (22) implies that the skin friction in the x direction has the same sign as b. Wilkinson took $\alpha > 0$. This is unnecessary for $\beta > \beta_*$ but is necessary for $\beta = \beta_*$, because if $\alpha < 0$ the expansion (22) is not real. These results can be clearly appreciated by considering the skin friction in α, β space: the double root for $\alpha > 0$ and the complex double root for $\alpha < 0$ are apparent.

5. A two-dimensional boundary layer on a surface of small curvature

We next find the effect of longitudinal surface curvature on the Falkner–Skan flow. Narasimha & Ojha (1967) and Cooke (1966) considered the perturbation of the Falkner–Skan solution due to small curvature of the wall. They assumed that the displacement surface is given such that the outer flow asymptotically approaches the potential flow which has longitudinal velocity $x^m\mathbf{i}$, and then sought to iterate the solution to the Navier–Stokes equations at large Reynolds numbers by matching inner and outer solutions. This led to the following problem (cf. Narasimha & Ojha 1967, equations (3.7)–(3.9), after correction of the misprinted sign of A in equation (3.7)).

$$Lf_1 = kR_1, \tag{36}$$

$$Bf_1 = -k \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}, \tag{37}$$

where B and L are as defined in (P_1), k is a constant proportional to the local curvature (which may be positive or negative), the function R_1 is given by

$$R_1(\eta) \equiv f_0''(\eta f_0 - 1) - f_0 f_0' - \beta\left\{\eta(f_0'^2 - 1) - \frac{2}{1+\beta}(f_0'' + f_0 f_0' + \beta\eta + A)\right\}, \tag{38}$$

the constant zero-order displacement thickness

$$A \equiv \lim_{\eta\to\infty} (\eta - f_0),$$

and f_0 is the solution of the Falkner–Skan equation for given β and $k = 0$. In fact the condition that $f_1' \sim -k\eta$ as $\eta \to \infty$ and equation (36) imply that

$$f_1' = -k\eta + o(1) \quad \text{as} \quad \eta \to \infty.$$

The method of solution of Narasimha & Ojha (1967) and of Cooke (1966) was essentially to fix β and formally expand the inner solution

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{39}$$

for small ϵ, where ϵ is the reciprocal of the square-root of the Reynolds number, with a similar expansion for the outer solution. Their numerical results indicate that the solution f_1 exists for $\beta_* < \beta \leqslant 2$ but not for $\beta = \beta_*$.

Our analysis confirms their conclusion as follows. A necessary condition for the existence of f_1 when $\beta = \beta_*$ is that

$$\begin{aligned} 0 &= k^{-1} \lim_{\eta\to\infty} \int_0^\eta f_1 L^\dagger h^\dagger \, d\eta \\ &= k^{-1} \lim_{\eta\to\infty} \left\{ \int_0^\eta h^\dagger L f_1 \, d\eta - [h^\dagger f_1'' - (h^{\dagger\prime} - f_0 h^\dagger) f_1' + \{h^{\dagger\prime\prime} - f_0 h^{\dagger\prime} - (1+2\beta_*) f_0' h^\dagger\} f_1]_0^\eta \right\} \\ &= \lim_{\eta\to\infty} \left\{ \int_0^\eta h^\dagger R_1 \, d\eta - \eta (h^{\dagger\prime} - f_0 h^\dagger) \right\} \\ &= K_1, \quad \text{say,} \end{aligned} \tag{40}$$

where
$$K_1 \equiv \int_0^\infty (R_1 + \eta f_0' + f_0) h^\dagger + \eta f_0 h^{\dagger\prime} \, d\eta. \tag{41}$$

It can be easily shown that $R_1 = (2\beta_* - 1)\eta + A + o(1)$ and that

$$h^\dagger = \text{constant} \times (\eta - A)^{-(1+2\beta_*)} + O(\eta - A)^{-(3+2\beta_*)} \quad \text{as} \quad \eta \to +\infty,$$

and thence that the integral converges. Although it converges algebraically rather than exponentially, the same numerical method as was used in § 2 gives $K_1 = 1{\cdot}103$ with accuracy to four significant figures. Thus $K_1 \neq 0$, condition (40) is violated and the solution f_1 of system (36) and (37) does not exist when $\beta = \beta_*$.

To cure this trouble, we formally expand

$$f = f_0(\eta) + \epsilon^{\frac{1}{2}} f_{\frac{1}{2}}(\eta) + \epsilon f_1(\eta) + \dots, \tag{42}$$

and proceed much as we did in § 2. This gives

$$L f_{\frac{1}{2}} = 0, \quad B f_{\frac{1}{2}} = 0, \tag{43}$$

and thence
$$f_{\frac{1}{2}} = c(-k)^{\frac{1}{2}} f_0'(\eta) \tag{44}$$

for some constant c to be determined. Therefore

$$L f_1 = k R_1 - c^2 k \{\beta_* f_0''^2 + f_0 f_0' f_0'' + \beta_* f_0' (1 - f_0'^2)\}, \tag{45}$$

and the adjoint relation gives

$$0 = k^{-1}\lim_{\eta\to\infty}\int_0^\eta f_1 L^\dagger h^\dagger\, d\eta = K_1 - c^2 I_2.$$

Therefore
$$c = \pm (K_1/I_2)^{\frac{1}{2}} = \pm 3{\cdot}330. \tag{46}$$

It can now be seen that, because K_1, $I_2 > 0$, a real solution is possible only if $k < 0$; this is why we took the square-root of $-k$ rather than of k in (44). The solution (42) gives
$$f = f_0 + c(-k\epsilon)^{\frac{1}{2}} f_0' + O(k\epsilon) \tag{47}$$
as $\epsilon \to 0$.

We note that for $k > 0$ (i.e. convex wall) there is no real solution but that for $k < 0$ (i.e. concave wall) there are two solutions corresponding to the alternative signs of c. There is an analogy between these results and those of § 4: for $\alpha > 0$ or $k < 0$ the basic Falkner–Skan flow is being accelerated and there are two solutions, while for $\alpha < 0$ or $k > 0$ the Falkner–Skan boundary layer is being further retarded so there is no solution, just as there is no solution to the Falkner–Skan problem for $\beta < \beta_*$.

This section indicates that at the onset of separation the correction to the profile is of the order of the inverse *fourth*-root of the Reynolds number rather than the usual inverse square-root of boundary-layer theory.

6. Instability of shear flow in a stratified fluid

For our final example, we consider a field of fluid mechanics outside boundary-layer theory, namely hydrodynamic instability. For plane parallel flows with antisymmetric velocity profiles it is often assumed that the phase velocity of each normal mode is zero. Justification of this assumption depends upon uniqueness of the mode as well as antisymmetry (cf. Drazin 1958, p. 220; Tatsumi & Gotoh 1960, p. 440), and uniqueness is seldom easy to prove. The velocity of a mode is often zero for some range of a parameter, but the mode may split into two modes with equal and opposite phase velocities as the parameter increases beyond this range; plane Couette flow offers a classic example of this, the modes bifurcating as the Reynolds number increases. Numerical work is simplified by the assumption that the phase velocity is zero, so analysis of the bifurcation of a mode may be very useful, showing where and how the phase velocity ceases to be zero.

These remarks apply to many problems of linear instability, but are well illustrated by a problem recently solved by Huppert (1973). Following Høiland & Riis (1968), he showed how the instability of a certain plane parallel flow of incompressible inviscid stratified fluid may be reduced to the following eigenvalue problem:
$$d^2\phi/dy^2 + \{Jy^2(y-c)^{-2} - \alpha^2\}\phi = 0, \tag{48}$$
$$\phi = 0 \quad \text{at} \quad y = \pm\pi. \tag{49}$$

One seeks to find the complex eigenvalue c and eigenfunction ϕ in terms of any given real wavenumber α and positive Richardson number J. Høiland & Riis (1968) noted two classes of eigensolutions with
$$c = c_0 \equiv 0, \quad \phi = \phi_0 \equiv \sin my, \quad \alpha^2 = \alpha_m^2 \equiv J - m^2, \tag{50}$$
$$c = c_0 \equiv 0, \quad \phi = \phi_0 \equiv \cos(n-\tfrac{1}{2})\,y, \quad \alpha^2 = \alpha_n^2 \equiv J - (n-\tfrac{1}{2})^2, \tag{51}$$

for positive integers $m < J^{\frac{1}{2}}$, $n < \frac{1}{2}+J^{\frac{1}{2}}$. Huppert proceeded to perturb the eigenvalues of these solutions, finding that

$$c \sim i\left\{\frac{2J^2}{m\pi}\operatorname{Cin}(2m\pi)\operatorname{Si}(2m\pi)-\frac{6Jm}{\pi}\operatorname{Si}(2m\pi)\right.$$
$$\left.+J^2\int_0^1 \cos 2m\pi t \log^2\left(\frac{t}{1-t}\right)dt\right\}^{-\frac{1}{2}}(\alpha^2-\alpha_m^2)^{\frac{1}{2}} \quad \text{as} \quad \alpha\to\alpha_m, \tag{52}$$

$$c \sim -\tfrac{1}{2}iJ^{-1}(\alpha^2-\alpha_n^2) \quad \text{as} \quad \alpha\to\alpha_n, \tag{53}$$

for fixed J, where the sine and cosine integrals are respectively defined by

$$\operatorname{Si}(z) \equiv \int_0^z \frac{\sin t}{t}\,dt, \quad \operatorname{Cin}(z) \equiv \int_0^z \frac{1-\cos t}{t}\,dt.$$

We perturb the solution (51) as follows, trying

$$\left.\begin{aligned} \phi &= \phi_0+(\alpha^2-\alpha_n^2)\,\phi_1+(\alpha^2-\alpha_n^2)^2\,\phi_2+\ldots,\\ c &= c_0+(\alpha^2-\alpha_n^2)\,c_1+\ldots. \end{aligned}\right\} \tag{54}$$

Equating coefficients of $\alpha^2-\alpha_n^2$, we get

$$\left.\begin{aligned} M\phi_1 &\equiv d^2\phi_1/dy^2+\{Jy^2/(y-c_0)^2-\alpha_n^2\}\,\phi_1\\ &= \{1-2c_1Jy^2/(y-c_0)^3\}\,\phi_0,\\ \phi_1 &= 0 \quad \text{at} \quad y=\pm\pi. \end{aligned}\right\} \tag{55}$$

Therefore

$$0 = \left[\phi_0\frac{d\phi_1}{dy}-\phi_1\frac{d\phi_0}{dy}\right]_{-\pi}^{\pi} = \int_{-\pi}^{\pi}\phi_0\frac{d^2\phi_1}{dy^2}-\phi_1\frac{d^2\phi_0}{dy^2}\,dy = \int_{-\pi}^{\pi}\phi_0 M\phi_1-\phi_1 M\phi_0\,dy$$
$$= \int_{-\pi}^{\pi}\left\{1-\frac{2c_1Jy^2}{(y-c_0)^3}\right\}\phi_0^2\,dy. \tag{56}$$

Therefore

$$c_1 = \int_{-\pi}^{\pi}\cos^2(n-\tfrac{1}{2})\,y\,dy\Big/J\left\{2\int_{-\pi}^{\pi}\frac{dy}{y}-\int_{-\pi}^{\pi}\frac{1-\cos(2n-1)\,y}{y}\,dy\right\}$$
$$= -\tfrac{1}{2}iJ^{-1}, \tag{57}$$

if the path of the singular integral is taken round the origin on the usual basis. This basis is justified by Howard (1963), and indeed our treatment is essentially equivalent to Huppert's calculation of $dc/d\alpha^2$ by Howard's method. However, solution (50) cannot be perturbed in this way, because it gives $0=\pi$ when substituted into (56). Thus expansion (54) is not valid for solution (50) when n is replaced by m.

Instead we try

$$\left.\begin{aligned} \phi &= \phi_0+(\alpha^2-\alpha_m^2)^{\frac{1}{2}}\,\phi_{\frac{1}{2}}+(\alpha^2-\alpha_m^2)\,\phi_1+\ldots,\\ c &= c_0+(\alpha^2-\alpha_m^2)^{\frac{1}{2}}\,c_{\frac{1}{2}}+(\alpha^2-\alpha_m^2)\,c_1+\ldots. \end{aligned}\right\} \tag{58}$$

This gives $$M\phi_{\frac{1}{2}} = -2Jc_{\frac{1}{2}}y^2\phi_0(y-c_0)^{-3},$$

i.e. $$\frac{d^2\phi_{\frac{1}{2}}}{dy^2}+m^2\phi_{\frac{1}{2}} = -\frac{2c_{\frac{1}{2}}J\sin my}{y} \tag{59}$$

and $$\phi_{\frac{1}{2}} = 0 \quad \text{at} \quad y=\pm\pi. \tag{60}$$

Therefore $$\phi_{\frac{1}{2}} = -2c_{\frac{1}{2}}JF(y), \tag{61}$$

where it can be shown by variation of parameters and a little calculus that

$$F = (2m)^{-1}\{\sin my\,\mathrm{Si}\,(2my) - \cos my\,\mathrm{Cin}\,(2my) + \cos my\,\mathrm{Cin}\,(2m\pi)\}. \qquad (62)$$

We could add to F any multiple of $\phi_0 = \sin my$, but this would merely be equivalent to renormalization of the solution ϕ of the linear problem. So we may choose F as above without loss of generality, this choice having the convenience of giving F as an even function of y.

The first-order problem is that

$$\left.\begin{aligned} M\phi_1 &= \left\{1 - \frac{2c_1 Jy^2}{(y-c_0)^3}\right\}\phi_0 - \frac{2c_{\frac{1}{2}} Jy^2}{(y-c_0)^3}\phi_{\frac{1}{2}} - \frac{3c_{\frac{1}{2}}^2 Jy^2}{(y-c_0)^4}\phi_0,\\ \phi_1 &= 0 \quad \text{at} \quad y = \pm\pi. \end{aligned}\right\} \qquad (63)$$

The solubility condition for ϕ_1 now gives

$$0 = \left[\phi_0 \frac{d\phi_1}{dy} - \phi_1 \frac{d\phi_0}{dy}\right]_{-\pi}^{\pi}$$

$$= \int_{-\pi}^{\pi} \phi_0^2\,dy + c_{\frac{1}{2}}^2\left\{4J^2\int_{-\pi}^{\pi}\frac{y^2\phi_0 F}{(y-c_0)^3}\,dy - 3J\int_{-\pi}^{\pi}\frac{y^2\phi_0^2}{(y-c_0)^4}\,dy\right\}. \qquad (64)$$

Therefore

$$c_{\frac{1}{2}}^2 = \left\{\frac{6J}{\pi}\int_0^{\pi}\frac{\sin^2 my}{y^2}\,dy - \frac{8J^2}{\pi}\int_0^{\pi}\frac{F\sin my}{y}\,dy\right\}^{-1}, \qquad (65)$$

so

$$c_{\frac{1}{2}} \sim i\left\{\frac{4J^2}{m\pi}\int_0^{2m\pi}\frac{1-\cos t}{t}\,\mathrm{Si}\,(t)\,dt - \frac{6Jm}{\pi}\,\mathrm{Si}\,(2m\pi)\right\}^{-\frac{1}{2}}(\alpha^2-\alpha_m^2)^{\frac{1}{2}} \quad \text{as} \quad \alpha \to \alpha_m. \qquad (66)$$

It can easily be shown that this asymptotic result is the same as Huppert's result (52) if and only if two integrals are identical, the integrals arising from the coefficients of J^2. We are grateful to Dr Susan Brown for proving the integral identity, by use of multiple integrals and change of variables. Taking an example, we found numerically that the results give $c_{\frac{1}{2}} = 0{\cdot}11394i$ for $J = 6$ and $m = 2$. This compares well with direct integration of (48), because computation of the eigenvalue for $J = 6$ gives $c = 0{\cdot}013892i$ when $\alpha = 1{\cdot}42$ and $c = 0{\cdot}005288i$ when $\alpha = 1{\cdot}415$. Fitting these two results to a quadratic of the form

$$c = c_{\frac{1}{2}}(\alpha^2-\alpha_m^2)^{\frac{1}{2}} + c_1(\alpha^2-\alpha_m^2),$$

we find that $c_{\frac{1}{2}} = 0{\cdot}1142i$ and $c_1 = -0{\cdot}0448i$.

The advantage of our method lies in both its relative simplicity and generality. It does not depend upon the luck of being able to solve the Taylor–Goldstein equation (48) in terms of confluent hypergeometric or other known functions but merely upon being able to invert the simple differential operator M associated with the solution to be perturbed. The method further shows clearly why an expansion in powers of $(\alpha^2-\alpha_m^2)^{\frac{1}{2}}$ rather than $\alpha^2-\alpha_m^2$ is necessary.

Finally note that we require $\alpha^2 > \alpha_m^2$ to validate our expansion, because otherwise (66) would give real values of c, so that the problem would be singular.

We have avoided discussion of the deeper topic of the singularity because it is not very relevant to this application of a series in powers of the square-root of $\alpha^2 - \alpha_m^2$.

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